

Gauge fixing by unitary transformations in QCD

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Abstract

The unitary gauge fixing technique is applied to the QCD hamiltonian formulated in terms of angular variables. It is demonstrated that in this formulation projections on the physical Hilbert space are unnecessary to separate physical and unphysical degrees of freedom. Therefore the application of the unitary gauge fixing technique can be extended to the operator level.

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The development of an analytical understanding of non-perturbative aspects of QCD is still one of the most challenging tasks. As a step towards this goal the interest in formulations of QCD, in which the gauge invariant variables are explicitly isolated, has been renewed recently. After the pioneering work of Goldstone and Jackiw [1] different such formulations have been derived [2]–[9] and on this basis an attempt to calculate approximately the spectrum of low lying states has been made [10]. Most of these aforementioned approaches are based on a particular ad hoc separation of gauge field degrees of freedom into gauge variant angular variables and gauge invariant variables. More general methods of eliminating unphysical variables, however, have also been developed [5, 9, 11]. They are particularly successful in QED, where unitary transformations within the physical subspace can be obtained, the property of which is to relate formulations in different gauges without using unphysical, gauge variant degrees of freedom [11]. In the application of the unitary gauge fixing technique to QCD, however, this generality could not be maintained. Instead explicit projections onto the physical subspace were necessary in order to derive one particular representation of the QCD hamiltonian, the axial gauge representation [9]. This presents an obstacle not only for establishing the aforementioned relation between different gauges, but also obscures the important issue of residual symmetries which has been linked to the presence of massless excitations [9, 12]. In this letter we want to demonstrate that these limitations do not arise, if the hamiltonian is written in terms of angular variables. Introducing angular degrees of freedom explicitly, allows us to perform the unitary gauge fixing transformations on the operator level. To demonstrate this possibility we reproduce the axial gauge representation of the QCD hamiltonian which was formulated in [9].

We use the hamiltonian formulation in which the dynamics is defined in terms of the standard canonically quantized hamiltonian in the $A_0 = 0$ gauge

$$\mathcal{H} = \sum_i \bar{\psi}(x) \gamma_i (i\partial_i + gA_i) \psi(x) + m\bar{\psi}(x)\psi(x) + \frac{1}{2} \sum_i E_i^a(x) E_i^a(x) + \frac{1}{4} \sum_{ij} F_{ij}^a F_{ij}^a \quad (1)$$

supplemented by the constraint that the generators of time independent gauge transformations, Gauss' law operators G^a , should annihilate physical states

$$G^a(x) = \sum_i \left[\partial_i E_i^a(x) + g f^{abc} A_i^b(x) E_i^c(x) \right] + g\rho(x); \quad G^a(x)|phys\rangle = 0. \quad (2)$$

In these expressions space is assumed to be a torus of volume L^d and the gauge fields are canonically quantized with periodic, fermion fields with anti-periodic boundary conditions. The gauge field may be written in SU(N) gauge theories as¹

$$A_i(x) = \frac{i}{g} V_i(x) \partial_i V_i^\dagger(x) \quad (\text{no summation}) \quad (3)$$

¹Note that we deviate from the usual convention in that spatial indices are summed over only, if explicitly indicated.

where the $SU(N)$ -matrices V_i are uniquely determined, given a definite set of paths which we choose to be straight lines parallel to one of the coordinate axes

$$V_i(x) = P \exp \left[ig \int_0^{x_i} dz_i A_i(x_i^\perp, z_i) \right] = \exp \left[i \xi_i^a(x) \frac{\lambda^a}{2} \right] . \quad (4)$$

P in this expression denotes path ordering and x_i^\perp stands for all coordinates orthogonal to x_i . The matrices $V_i(x)$ are furthermore parametrized in terms of "angles" $\xi_i(x)$. In order to formulate the hamiltonian using this new set of variables we also introduce angular momentum operators $J_k^c(x)$ with commutation relations

$$\begin{aligned} [J_i^a(x), V_j(y)] &= \delta_{i,j} \delta^d(x-y) V_j(y) \frac{\lambda^a}{2} \\ [J_i^a(x), J_j^b(y)] &= \delta_{i,j} i f^{abc} J_i^c(x) \delta^d(x-y) \end{aligned} \quad (5)$$

in terms of which the electric field operator can be written as

$$\begin{aligned} E_i^a(x) &= g \int d^d z \delta^{d-1}(z_i^\perp - x_i^\perp) \theta(z_i - x_i) \theta(x_i) N_i^{ac}(x) J_i^c(z) \\ N_i^{ac}(x) &= \text{Tr} \left\{ V_i(x) \frac{\lambda^a}{2} V_i^\dagger(x) \lambda^c \right\} . \end{aligned}$$

Using these relations one easily rewrites the hamiltonian in the form (an analogous expression was derived in [13])

$$\begin{aligned} \mathcal{H} &= \sum_i \left[\bar{\psi}(x) V_i(x) \right] \gamma_i i \partial_i \left[V_i^\dagger(x) \psi(x) \right] + m \bar{\psi}(x) \psi(x) \\ &+ \frac{g^2}{2} \sum_i \int_{x_i}^L dz_i J_i^b(x_i^\perp, z_i) \int_{x_i}^L dz'_i J_i^b(x_i^\perp, z'_i) \\ &+ \frac{1}{2g^2} \sum_{ij} \text{Tr} \left\{ \left[\partial_i \left[(V_i^\dagger V_j) \partial_j (V_j^\dagger V_i) \right] \right] \left[\partial_i \left[(V_i^\dagger V_j) \partial_j (V_j^\dagger V_i) \right] \right]^\dagger \right\} \end{aligned} \quad (6)$$

and for the Gauss' law operators we obtain the following expressions

$$G^a(x) = -g \sum_i \left[N_i^{ac}(x) J_i^c(x) - \delta(x_i) \int dz_i J_i^a(x_i^\perp, z_i) \right] + g \rho^a(x) . \quad (7)$$

The technique to use unitary transformations in order to eliminate unphysical degrees of freedom [9, 11] is, in the case of the axial gauge $A_3^a = 0$, based on the use of the transverse Gauss' law operator G_\perp in terms of which the following unitary transformation T was constructed in [9]

$$G_\perp^a(x) = -g \sum_{i \neq 3} \left[N_i^{ac}(x) J_i^c(x) - \delta(x_i) \int dz_i J_i^a(x_i^\perp, z_i) \right] + g \rho^a(x) \quad (8)$$

$$T = \exp \left[\frac{i}{g} \int d^d z G_{\perp}^{a_0}(z) \theta^{a_0}(z_{\perp}^{\perp}) \frac{z_3}{L} \right] \exp \left[\frac{-i}{g} \int d^d z G_{\perp}^a(z) \Delta^a(z_{\perp}^{\perp}) \right] \quad (9)$$

$$\exp \left[\frac{-i}{g} \int d^d z G_{\perp}^a(z) \xi_3^a(z) \right] . \quad (10)$$

$\Delta(x_3^{\perp})$ and $\theta(x_3^{\perp})$ in these expressions are defined by diagonalizing $\xi_3(x_3^{\perp}, L)^2$

$$\exp \left[-i \Delta(x_3^{\perp}) \right] \exp \left[i \xi_3(x_3^{\perp}, L) \right] \exp \left[i \Delta(x_3^{\perp}) \right] = \exp \left[i \theta(x_3^{\perp}, L) \right] \quad (11)$$

$$\exp \left[-i \Delta(x_3^{\perp}) \right] \frac{\lambda^a}{2} \exp \left[i \Delta(x_3^{\perp}) \right] = \sum_b R_{ab}(x_3^{\perp}) \frac{\lambda^b}{2} \quad (12)$$

$$\text{Tr} \left\{ e^{-i \theta(x_3^{\perp})} \frac{\lambda^{a_1}}{2} e^{i \theta(x_3^{\perp})} \lambda^{b_1} \right\} - \delta_{a_1 b_1} = P_{a_1 b_1}(x_3^{\perp}) . \quad (13)$$

For later use we have already introduced the matrices $R(x_3^{\perp})$ and $P(x_3^{\perp})$. Taking notice of the following general relation which is valid for any function φ of the "angles" $\xi_3(x)$

$$\begin{aligned} \exp \left[\frac{-i}{g} \int d^d z G_{\perp}^a(z) \varphi^a(z) \right] \frac{-i \delta}{\delta \xi_3^b(x)} \exp \left[\frac{i}{g} \int d^d z G_{\perp}^a(z) \varphi^a(z) \right] = \\ \frac{-i \delta}{\delta \xi_3^b(x)} + \frac{2}{g} \int d^d z \text{Tr} \left\{ G_{\perp}(z) e^{-i \varphi(z)} \frac{-i \delta}{\delta \xi_3^b(x)} e^{i \varphi(z)} \right\} \end{aligned} \quad (14)$$

we can unitarily transform $J_3^a(x)$ and $G^a(x)^3$. We find the following result

$$\begin{aligned} T g J_3^a(x) T^{\dagger} &= g J_3^a(x) + R_{ac_0}(x_3^{\perp}) \left[G_{\perp}^{c_0}(x) - \delta(x_3 - L) \int dz \frac{z}{L} G_{\perp}^{c_0}(x_3^{\perp}, z) \right] \\ &\quad + R_{ad_1}(x_3^{\perp}) \left[\mathcal{G}^{d_1}(x) + \delta(x_3 - L) P_{d_1 c_1}^{-1}(x_3^{\perp}) \int dz \mathcal{G}^{c_1}(x_3^{\perp}, z) \right] \\ T G^a(x) T^{\dagger} &= -g N_3^{ac}(x) J_3^c(x) + \delta(x_3) \int dz \left[g J_3^a(x_3^{\perp}, z) + R_{ac_0}(x_3^{\perp}) G_{\perp}^{c_0}(x_3^{\perp}, z) \right] . \end{aligned} \quad (15)$$

Upon decomposing $J_3^a(x)$ with the use of a δ -function in the following way

$$J_3^a(x) = \frac{1}{L} \sum_n J_3^a(x_3^{\perp}, n) \left[e^{i 2 \pi n x_3 / L} - 1 \right] + \frac{1}{L} \sum_n e^{i 2 \pi n x_3 / L} J_3^a(x_3^{\perp}, L) . \quad (16)$$

one can separate the Gauss' law constraints into three independent contributions

$$0 < x_3 < L : J_3^a(x) |phys\rangle = 0 \quad (17)$$

$$R_{ab_1}(x_3^{\perp}) J_3^a(x_3^{\perp}, L) |phys\rangle = 0 \quad (18)$$

$$\int_0^L dz G_{\perp}^{b_0}(x_3^{\perp}, z) |phys\rangle = 0 \quad (19)$$

²Note that we adopt the convention that indices with subindex 0 enumerate the elements of the Cartan subalgebra of $SU(N)$ and indices with subindex 1 enumerate all remaining elements.

³Exept these two all variables associated with $A_{i \neq 3}$ are, by construction, just gauge transformed with the transformation function depending on $\xi_3(x)$.

the last of which eq.(19) generates residual abelian gauge transformations. These can be implemented without difficulty [9] and for this reason they shall not be considered here any further.

Having identified the independent constraints, the hamiltonian can be decomposed into a part which is proportional to Gauss' law operators and thus vanishes on physical states and a remaining hamiltonian acting non-trivially in the physical Hilbert space. For the latter part we find using the identities given in the appendix

$$\begin{aligned}
& T \left[\int dx_3 \int_{x_3}^L dy \int_{x_3}^L dz \frac{g^2}{2} J_3^a(x_3^\perp, y) J_3^a(x_3^\perp, z) \right] T^\dagger \stackrel{phys.space}{=} \frac{g^2 L}{2} J_3^a(x_3^\perp, L) J_3^a(x_3^\perp, L) \quad (20) \\
& + \frac{1}{2} \int dy \int dz [\theta(y-z)z + \theta(z-y)y] \mathcal{G}^{c_1}(x_3^\perp, y) \mathcal{G}^{c_1}(x_3^\perp, z) \\
& + \frac{L}{2} P_{c_1 a_1}^{-1}(x_3^\perp) \int dy \int dz \left[\left(P_{c_1 b_1}^{-1}(x_3^\perp) + \frac{y}{L} \delta_{c_1, b_1} \right) \delta_{a_1, d_1} + \frac{z}{L} \delta_{c_1, d_1} \delta_{a_1, b_1} \right] \mathcal{G}^{b_1}(x_3^\perp, y) \mathcal{G}^{d_1}(x_3^\perp, z) \\
& - \frac{1}{2} \int dy \int dz \left[\frac{yz}{L} - \theta(y-z)z - \theta(z-y)y \right] G_\perp^{c_0}(x_3^\perp, y) G_\perp^{c_0}(x_3^\perp, z) \quad (21)
\end{aligned}$$

where we note that the contribution $J_3^a(x_3^\perp, L) J_3^a(x_3^\perp, L)$ has not yet been reduced to the relevant part in the physical Hilbert space since eq.(18) has not yet been used. The contribution in the physical subspace depending on G_\perp can be brought into the following form

$$\begin{aligned}
H_{int} &= \int d^{d-1} x_3^\perp \int dy \int dz \left[-\frac{|y-z|}{4} + \frac{(y-z)^2}{4L} \right] G_\perp^{c_0}(x_3^\perp, y) G_\perp^{c_0}(x_3^\perp, z) \\
&+ \frac{L}{2} \int d^{d-1} x_3^\perp \int dy \int dz \sum_{nm} \mathcal{G}_{nm}(x_3^\perp, y) \mathcal{G}_{mn}(x_3^\perp, z) \left[\sin^{-2} \left[\frac{\mu_n(x_3^\perp) - \mu_m(x_3^\perp)}{2} \right] \right. \\
&\quad \left. + i \frac{2}{L} \cot \left[\frac{\mu_n(x_3^\perp) - \mu_m(x_3^\perp)}{2} \right] (y-z) - \frac{2}{L} |y-z| \right] . \quad (22)
\end{aligned}$$

The μ_n in these expressions are defined in eq.(29) and the matrices \mathcal{G} are related to the perpendicular Gauss' law operators in the following way

$$\mathcal{G}_{nm}(x) = G_\perp^{c_1}(x) \frac{\lambda_{nm}^{c_1}}{2} e^{-i(\mu_n - \mu_m)x_3/L}$$

which shows that H_{int} contains singularities only at those points x_3^\perp where $\mu_n(x_3^\perp)$ and $\mu_m(x_3^\perp)$ coincide for some $n \neq m$. Using eq.(18) and the identities given in the appendix we eventually find in agreement with the results in [9] for the hamiltonian in the physical Hilbert space the form

$$H = \int d^d x \left\{ \sum_{i \neq 3} [\bar{\psi}(x) V_i(x)] \gamma_i i \partial_i [V_i^\dagger(x) \psi(x)] + [\bar{\psi}(x) e^{i\theta(x_3^\perp) \frac{x_3}{L}}] \gamma_3 i \partial_3 [e^{-i\theta(x_3^\perp) \frac{x_3}{L}} \psi(x)] \right\}$$

$$\begin{aligned}
& + \frac{g^2}{2} \sum_{i \neq 3} \int_{x_i}^L dz_i J_i^b(x_i^\perp, z_i) \int_{x_i}^L dz'_i J_i^b(x_i^\perp, z'_i) + \mathcal{K}_3(x_3^\perp) \\
& + \frac{1}{2} \sum_{i,j \neq 3} \text{Tr} \{ F_{ij}(x) F_{ij}(x) \} + \sum_{i \neq 3} \text{Tr} \{ \tilde{F}_{i3}(x) \tilde{F}_{i3}(x) \} + m \bar{\psi}(x) \psi(x) \Big\} + H_{int} .
\end{aligned} \tag{23}$$

H_{int} has been given in eq.(22) and \mathcal{K}_3 denotes the kinetic energy operator for the fields $\theta^{c0}(x_3^\perp)$ which is of the form derived in eq.(30)

$$\begin{aligned}
\mathcal{K}_3(x_3^\perp) &= \prod_{k < l} \sin^{-2} \frac{1}{2} \left[\mu_k(x_3^\perp) - \mu_l(x_3^\perp) \right] \frac{-i\delta}{\delta\theta^{c0}(x_3^\perp)} \\
&\cdot \prod_{n < m} \sin^2 \frac{1}{2} \left[\mu_n(x_3^\perp) - \mu_m(x_3^\perp) \right] \frac{-i\delta}{\delta\theta^{c0}(x_3^\perp)} .
\end{aligned} \tag{24}$$

Finally \tilde{F}_{i3} in the hamiltonian is obtained from F_{i3} by the replacement

$$A_3(x) \rightarrow \frac{i}{g} e^{i\theta(x_3^\perp)x_3/L} \partial_3 e^{-i\theta(x_3^\perp)x_3/L} . \tag{25}$$

This result shows that all non-trivial properties, including the singular 'Coulomb'-interaction and the Jacobian for the variables θ^{c0} are correctly reproduced. Moreover all operations necessary for this purpose can be performed on the operator level. Projections onto the physical subspace have been used in formulating eq.(23) but exclusively in order to obtain simple expressions in which irrelevant terms do not show up anymore. At no stage in the derivation this projection was essential. As discussed in detail in [9, 11] this offers the possibility to transform the hamiltonian within the physical subspace from one representation to another. It may also help to clarify the properties of residual gauge transformations and their relation to the Gribov problem.

Furthermore one can use the formulation in terms of angular variables together with the technique of unitary gauge fixing transformations to investigate new gauges. This may allow one to find an optimal starting point for approximating the full dynamics and thus for extracting physical information on the low energy properties of QCD. This question is presently under investigation.

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A Useful identities for the unitary transformations

In this appendix we give useful identities for the derivation of the hamiltonian in the physical Hilbert space eq.(23). For the commutators of J_3 with \mathcal{G} we find

$$[J_3^a(x_3^\perp, L), \mathcal{G}^{c1}(z)] = \delta^{d-1}(x_3^\perp - z_3^\perp) i f^{b_0 c_1 b_1} R_{ab_0}(x_3^\perp) \mathcal{G}^{b_1}(z) \frac{z_3}{L} \tag{26}$$

and with the matrices R and P the following commutation relations are obtained

$$\begin{aligned} \left[J_3^a(x_3^\perp, L), R_{bc}(z_3^\perp) \right] &= \delta^{d-1}(x_3^\perp - z_3^\perp) i f^{c_1 c d} R_{bd}(x_3^\perp) R_{ab_1}(x_3^\perp) P_{b_1 c_1}^{-1}(x_3^\perp) \\ \left[J_3^a(x_3^\perp, L), P_{c_1 d_1}^{-1}(z_3^\perp) \right] &= -\delta^{d-1}(x_3^\perp - z_3^\perp) i f^{b_0 b_1 e_1} R_{ab_0}(x_3^\perp) P_{b_1 d_1}^{-1}(x_3^\perp) \left[\delta_{c_1, e_1} + P_{c_1, e_1}^{-1}(x_3^\perp) \right]. \end{aligned}$$

To isolate the physical part of the operator $J_3^a J_3^a$ we note that the constraint eq.(18) suggests that $R_{ab_0} J_3^a$, which is not subject of this constraint, may be used as a derivative operator with respect to θ^{b_0} . This is verified by observing the identities

$$R_{c_0 a}^{-1}(x_3^\perp) \left[J_3^a(x_3^\perp, L), e^{i\theta(z_3^\perp)} \right] = \delta^{d-1}(x_3^\perp - z_3^\perp) e^{i\theta(z_3^\perp)} \frac{\lambda^{c_0}}{2} \quad (27)$$

$$\begin{aligned} R_{c_0 a}^{-1}(x_3^\perp) \left[J_3^a(x_3^\perp, L), e^{i\Delta(z_3^\perp)} \right] &= 0 \\ \Rightarrow R_{c_0 a}^{-1}(x_3^\perp) J_3^a(x_3^\perp, L) &= \frac{-i\delta}{\delta\theta^{c_0}(x_3^\perp)}. \end{aligned} \quad (28)$$

Introducing the following notation for θ^{c_0} using cartesian unit vecors \vec{e}_n in N dimensions

$$\sum_{c_0} \theta^{c_0}(x_3^\perp) \frac{\lambda^{c_0}}{2} = \sum_{n=1}^N \mu_n(x_3^\perp) \vec{e}_n \vec{e}_n^\dagger \quad (29)$$

we finally obtain as part of the contribution of the electric field E_3 to the hamiltonian

$$\begin{aligned} J_3^a(x_3^\perp, L) J_3^a(x_3^\perp, L) &= R_{c_1 b}^{-1}(x_3^\perp) J_3^b(x_3^\perp, L) R_{c_1 c}^{-1}(x_3^\perp) J_3^c(x_3^\perp, L) \\ &\quad - \frac{\delta^2}{\delta\theta^{c_0}(x_3^\perp) \delta\theta^{c_0}(x_3^\perp)} \\ &\quad - \frac{\delta^{d-1}(0)}{2} \sum_{n < m} \cot \left[\frac{\mu_n(x_3^\perp) - \mu_m(x_3^\perp)}{2} \right] (\lambda_{nn}^{c_0} - \lambda_{mm}^{c_0}) \frac{\delta}{\delta\theta^{c_0}(x_3^\perp)} \end{aligned} \quad (30)$$

which gives rise to the operator \mathcal{K}_3 in the hamiltonian acting on the physical Hilbert space eq.(23).

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